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NECESSARY CONDITIONS FOR THE EXISTENCE OF GLOBAL SOLUTIONS TO SYSTEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Paul L. Butzer
on the occasion of his 80th birthday

Abstract

Necessary conditions for the existence of global solutions to certain 2×2-systems of fractional differential equations are presented. It is demonstrated that fractional derivatives of lower order have a strong influence on the character of the solutions. Our method of proof relies on a suitable choice of the test function in the weak formulation of the solutions to the system under study.

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1. Introduction

Recent studies, see for example [5, 6, 7] and their extensive list of references (articles and books), demonstrate the role played by fractional derivatives in the mathematical modeling of various practical situations in mechanics, physical chemistry, biology and finance.

In this article, we want to present exponents threshold for the existence of global solutions to the system of equations containing fractional derivatives

$$\left\{ \begin{array}{lcllcl} a\,u'(t) & + & b\,D^{\alpha}_{0+}(u-u_0) & = & f(t)\,|v|^q + F(t), & t>0 \\ c\,v'(t) & + & d\,D^{\beta}_{0+}(v-v_0) & = & g(t)\,|u|^p + G(t), & t>0 \end{array} \right. \tag{1}$$

subject to the initial conditions

$$u(0) = u_0, \qquad v(0) = v_0, \tag{2}$$

where $1 < p, q$, $0 < \alpha, \beta < 2$, a, b, c, d are constants, the functions $f(t), g(t)$ are positive functions for $t > 0$, while $F(t)$ and $G(t)$ are given functions with nonnegative averages. There is a number of interesting articles dealing with local existence, existence of global solutions and their behavior, blowing-up solutions, estimates of the interval of existence, and lower-bounds of solutions for various equations of Abel or Volterra type that are connected to fractional differential equations, see [1, 2, 3, 4, 5, 8, 9]. For systems of fractional differential equations we are concerning, the interesting dissertation [8] deals with local and global existence under natural conditions (sub-linear growth of the nonlinearities).

The type of the nonlinearities in system (1) appears in systems describing processes of heat diffusion and combustion in two component continua with nonlinear heat conductance and volumetric release (they read $u_t = \Delta u + v^q$, $v_t = \Delta v + u^p$, the sub-script t stands for the time derivative, while Δ is the Laplace operator). So, the chosen nonlinearities may be viewed as a prototype of nonlinearities. Other choice could be $u^r v^q$ and $u^p v^s$ or e^v and e^u.

By choosing $a = c = f(t) = g(t) = 1$, $F(t) = G(t) = 0$, and $b = d = 0$ in (1) we obtain the particular system

$$\left\{ \begin{array}{lcll} u'(t) & = & |v|^q, & t>0 \\ v'(t) & = & |u|^p, & t>0 \end{array} \right. . \tag{3}$$

This system has, for $pq > 1$, the solution

$$u(t) = C_1 (T^{max} - t)^{-(q+1)/(pq-1)},$$
$$v(t) = C_2 (T^{max} - t)^{-(p+1)/(pq-1)},$$

for $0 < t < T^{max} < \infty$, where

$$C_1 = \left[(p+1)^q (q+1)/(pq-1)^{q+1}\right]^{1/(pq-1)},$$
$$C_2 = \left[(p+1)(q+1)^p/(pq-1)^{p+1}\right]^{1/(pq-1)}.$$

For $pq > 1$ the solution blows-up in the finite time

$$T^{max} = \left(\frac{C_2}{C_1} \frac{u_0}{v_0} \right)^{\frac{q-p}{pq-1}}.$$

However, the particular system containing only fractional derivatives,

$$\begin{cases} D^{\alpha}_{0+}(u - u_0) &= |v|^q, \quad t > 0, \\ D^{\beta}_{0+}(v - v_0) &= |u|^p, \quad t > 0, \end{cases} \tag{4}$$

has, for $pq > 1$ the blowing-up solution (due to Trujillo [11] in case of a single equation)

$$u(t) = C_{1,\alpha,\beta}(T^{max}_{\alpha,\beta} - t)^{-(\alpha+q\beta)/(pq-1)},$$
$$v(t) = C_{2,\alpha,\beta}(T^{max}_{\alpha,\beta} - t)^{-(\beta+p\alpha)/(pq-1)}$$

for $0 < t < T^{max}_{\alpha,\beta} < \infty$, where

$$C_{1,\alpha,\beta} = \left(\frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\beta)} \right)^{\frac{q}{pq-1}} \left(\frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1-\alpha)} \right)^{\frac{1}{pq-1}},$$

and

$$C_{2,\alpha,\beta} = \left(\frac{\Gamma(\delta+1)}{\Gamma(\delta+1-\beta)} \right)^{\frac{1}{pq-1}} \left(\frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1-\alpha)} \right)^{\frac{p}{pq-1}},$$

with

$$\delta = -\frac{p\alpha + \beta}{pq-1}, \qquad \varrho = -\frac{q\beta + \alpha}{pq-1}.$$

The blow-up time for this system is

$$T^{max}_{\alpha,\beta} = \left(\frac{C_{2,\alpha,\beta}}{C_{1,\alpha,\beta}} \frac{u_0}{v_0} \right)^{\frac{\alpha(1-p)-\beta(1-q)}{pq-1}},$$

which tends, when $\alpha \to 1,\ \beta \to 1$, to T^{max}.

For the system (1), no explicit solutions are available. In this paper we present a result concerning necessary conditions of existence.

The proof is based on a suitable choice of the test function in the weak formulation of the problem.

2. Preliminaries

We start by introducing some definitions and preliminary results needed for the principle result. For $0 < \alpha < 1$, we define the left-handed and right-handed fractional derivatives in the Riemann-Liouville sense by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\sigma)}{(t-\sigma)^{\alpha}}\, d\sigma,$$

and

$$D_{T-}^{\alpha} f(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{f(\sigma)}{(\sigma-t)^{\alpha}}\, d\sigma, \tag{5}$$

respectively.

It is shown in [10], Lemma 2.2, that for $f \in AC([0,T])$,

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(0)}{t^{\alpha}} + \int_0^t \frac{f'(\sigma)}{(t-\sigma)^{\alpha}}\, d\sigma \right]. \tag{6}$$

See also [7]. Note that

$$D_{0+}^{\alpha}(f - f(0))(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\sigma)}{(t-\sigma)^{\alpha}}\, d\sigma =: {}^{c}D_{0+}^{\alpha} f,$$

which is the Caputo derivative of f. For the right-hand derivative, we have

$$D_{T-}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^{\alpha}}\, d\sigma \right]. \tag{7}$$

It is shown in ([10], Corollary 2, p.46) that for functions $f, g \in C([0,t])$ with $D_{0+}^{\alpha} g$, $D_{T-}^{\alpha} f$ exist at every point in $[0,T]$ and are continuous, we have

$$\int_0^T f(t)\, D_{0+}^{\alpha} g(t)\, dt = \int_0^T D_{T-}^{\alpha} f(t)\, g(t)\, dt,$$

which is a formula for integration by parts.

To announce the preparatory lemmas, let

$$\varphi(t) = \begin{cases} (1 - t/T)^{\lambda}, & 0 \le t \le T, \\ 0, & t > T. \end{cases} \qquad \lambda \ge 2. \tag{8}$$

Then we have the following

Lemma 1. *Let $\varphi(t)$ be as in (8). Then*

$$\int_0^T D_{T-}^{\alpha} \varphi(t)\, dt = K_{\alpha,\lambda}\, T^{1-\alpha}, \quad K_{\alpha,\lambda} = \frac{\lambda \Gamma(\lambda-\alpha)}{(\lambda-\alpha+1)\, \Gamma(\lambda-2\alpha+1)}. \tag{9}$$

P r o o f. Using (5), we compute

$$D_{T-}^{\alpha} \varphi(t) = \frac{\lambda}{\Gamma(1-\alpha)}\, T^{-\lambda} \int_t^T \frac{(T-\sigma)^{\lambda-1}}{(\sigma-t)^{\alpha}}\, d\sigma.$$

Using the Euler change of variable

$$y = \frac{\sigma - t}{T - t},$$

we can write

$$D^{\alpha}_{T-}\varphi(t) = \frac{\lambda}{\Gamma(1-\alpha)}\, T^{-\lambda}\, (T-t)^{\lambda-\alpha} \int_0^1 \frac{(1-y)^{\lambda-\alpha-1}}{y^{\alpha}}\, dy.$$

The last integral is the Beta function of Euler:

$$\int_0^1 \frac{(1-y)^{\lambda-\alpha-1}}{y^{\alpha}}\, dy = \frac{\Gamma(\lambda-\alpha)\Gamma(1-\alpha)}{\Gamma(\lambda-2\alpha+1)}.$$

Thus,

$$D^{\alpha}_{T-}\varphi(t) = \frac{\lambda\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)}\, T^{-\lambda}(T-t)^{\lambda-\alpha}. \tag{10}$$

By integration we obtain the result. ■

Lemma 2. *Let φ be as in (8) and $p > 1$. Then for $\lambda > p-1$,*

$$\int_0^T \varphi^{1-p}(t)\, |\varphi'(t)|^p\, dt = \Lambda_p\, T^{1-p}, \tag{11}$$

where

$$\Lambda_p = \frac{\lambda^p}{\lambda+1-p}\,.$$

For $\lambda > \alpha p - 1$,

$$\begin{aligned} &\int_0^T \varphi^{1-p}(t)\, |D^{\alpha}_{T-}\varphi(t)|^p = \Lambda_{p,\alpha} T^{1-\alpha p}, \\ &\Lambda_{p,\alpha} = \tfrac{\lambda^p}{\lambda+1-p\alpha}\, \left[\tfrac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda+1-2\alpha)}\right]^p. \end{aligned} \tag{12}$$

P r o o f. Using (8), we have

$$\int_0^T \varphi^{1-p}(t)\, |\varphi'(t)|^p\, dt = \int_0^T \left[T^{-\lambda}(T-t)^{\lambda}\right]^{1-p} \left|-\lambda T^{-\lambda}(T-t)^{\lambda-1}\right|^p\, dt$$

$$= \lambda^p T^{-\lambda(1-p)} T^{-\lambda p} \int_0^T (T-t)^{\lambda(1-p)+(\lambda-1)p}\, dt$$

$$= \lambda^p T^{-\lambda} \int_0^T (T-t)^{\lambda-p}\, dt = \lambda^p T^{-\lambda} \frac{T^{\lambda+1-p}}{\lambda+1-p} = \frac{\lambda^p}{\lambda+1-p} T^{1-p}.$$

Using (8) and (10), we compute

$$\int_0^T \varphi^{1-p}(t)\, |D^{\alpha}_{T-}\varphi(t)|^p\, dt$$

$$= \int_0^T \left[T^{-\lambda}(T-t)^{\lambda}\right]^{1-p} \left|\frac{\lambda\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)}\, T^{-\lambda}(T-t)^{\lambda-\alpha}\right|^p\, dt$$

$$= T^{-\lambda(1-p)} \left[\frac{\lambda\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)} T^{-\lambda} \right]^p \int_0^T (T-t)^{\lambda(1-p)+p(\lambda-\alpha)}\, dt$$

$$= \left[\frac{\lambda\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)} \right]^p T^{-\lambda} \int_0^T (T-t)^{\lambda-p\alpha}\, dt \left[\frac{\lambda\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)} \right]^p T^{-\lambda} \frac{T^{\lambda+1-p\alpha}}{\lambda+1-p\alpha}.$$

■

REMARK 1. Note that the bound in (11) is a limiting case of the bound in (12) as $\alpha \to 1$. Moreover, since $\Gamma(\zeta)$ is increasing for $\zeta \geq 2$, for $\lambda \geq 3$ we have

$$\Lambda_{p,\alpha} < \Lambda_p.$$

For the rest of this paper we assume

$$\lambda > \max\{3, p-1, q-1\}. \tag{13}$$

For clarity, we consider first the case $a = b = c = d = 1$ (this is possible by a scaling argument), $f(t) = g(t) = 1$, $F(t) = G(t) = 0$ in (1). We will comment on this choice later. The reduced system we have, is:

$$\begin{cases} u'(t) + D^{\alpha}_{0+}(u-u_0) = |v|^q, & t > 0, \\ v'(t) + D^{\beta}_{0+}(v-v_0) = |u|^p, & t > 0. \end{cases} \tag{14}$$

It is classical to show (see [8]) that system (14) subject to conditions (2) has a solution

$$(u,v) \in \left(C^1(0,T_{max}) \cap C([0,T_{max})) \right)^2$$

with the life span T_{max} such that $T_{max} + \lim\limits_{t \to T_{max}} \{|u(t)| + |v(t)|\} = +\infty$.

In what follows, we treat separately the two cases $0 < \alpha, \beta < 1$ and $1 < \alpha, \beta < 2$.

3. The case $0 < \alpha, \beta < 1$

3.1. The main result

The main result is given by the following theorem.

THEOREM 1. *Let $1 < p, q$ and $u_0 > 0$, $v_0 > 0$, then if*

$$1 - \frac{1}{pq} \leq \alpha + \frac{\beta}{p}, \quad \text{or} \quad 1 - \frac{1}{pq} \leq \beta + \frac{\alpha}{q}, \tag{15}$$

the system (14) subject to (2) admits no global solutions.

P r o o f. Our proof is by contradiction. Let (u,v) be a global solution and φ as in (8) with the restriction (13) on λ. It is easy to see that (u,v) satisfies

$$\int_0^T |v|^q \varphi + u_0 \left(1 + \int_0^T D^\alpha_{T-}\varphi\right) = -\int_0^T u\,\varphi' + \int_0^T u\, D^\alpha_{T-}\varphi, \tag{16}$$

$$\int_0^T |u|^p \varphi + v_0 \left(1 + \int_0^T D^\beta_{T-}\varphi\right) = -\int_0^T v\,\varphi' + \int_0^T v\, D^\beta_{T-}\varphi. \tag{17}$$

Using the Hölder inequality, we have

$$\int_0^T u\,\varphi' = \int_0^T u\,\varphi^{1/p}\,\varphi^{-1/p}\,\varphi' \le \left(\int_0^T |u|^p \varphi\right)^{1/p} \left(\int_0^T \varphi^{-p'/p}\,|\varphi'|^{p'}\right)^{1/p'}, \tag{18}$$

$$\int_0^T v\,\varphi' \le \left(\int_0^T |v|^q \varphi\right)^{1/q} \left(\int_0^T \varphi^{-q'/q}\,|\varphi'|^{q'}\right)^{1/q'}, \tag{19}$$

$$\int_0^T u\, D^\alpha_{T-}\varphi \le \left(\int_0^T |u|^p \varphi\right)^{1/p} \left(\int_0^T \varphi^{-p'/p}\,|D^\alpha_{T-}\,\varphi|^{p'}\right)^{1/p'}, \tag{20}$$

and

$$\int_0^T v\, D^\beta_{T-}\varphi \le \left(\int_0^T |v|^q \varphi\right)^{1/q} \left(\int_0^T \varphi^{-q'/q}\,|D^\beta_{T-}\,\varphi|^{q'}\right)^{1/q'}, \tag{21}$$

where $p + p' = pp'$ and $q + q' = qq'$.

Let

$$\begin{aligned}
\mathcal{I} &:= \int_0^T |u|^p\,\varphi, & \mathcal{J} &:= \int_0^T |v|^q\,\varphi,\\
\mathcal{A} &:= \int_0^T \varphi^{-p'/p}\,|\varphi'|^{p'}, & \mathcal{B} &:= \int_0^T \varphi^{-q'/q}\,|\varphi'|^{q'},\\
\mathcal{C} &:= \int_0^T \varphi^{-p'/p}\,|D^\alpha_{T-}\,\varphi|^{p'}, & \mathcal{D} &:= \int_0^T \varphi^{-q'/q}\,|D^\beta_{t|T}\,\varphi|^{q'}.
\end{aligned}$$

We use (18), (19), (20) and (21), to write (16) and (17) in the forms

$$\mathcal{J} + u_0 \left(1 + \int_0^T D^\alpha_{T-}\varphi\right) \le \mathcal{I}^{1/p}\left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'}\right), \tag{22}$$

and

$$\mathcal{I} + v_0 \left(1 + \int_0^T D^\beta_{T-}\varphi\right) \le \mathcal{J}^{1/q}\left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right). \tag{23}$$

Using inequalities (22) and (23), we obtain the inequalities

$$\mathcal{J} \leq \mathcal{I}^{1/p} \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'} \right), \tag{24}$$

and

$$\mathcal{I} \leq \mathcal{J}^{1/q} \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'} \right), \tag{25}$$

from which we have

$$\mathcal{J}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'} \right) \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'} \right)^{1/p}, \tag{26}$$

and

$$\mathcal{I}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{1/p'} + \mathcal{C}^{1/p'} \right)^{1/q} \left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'} \right). \tag{27}$$

Using the elementary inequality

$$(x+y)^{1/r} \leq x^{1/r} + y^{1/r}, \quad r \geq 1, \; x > 0, y > 0,$$

we estimate the inequalities (26) and (27) as

$$\mathcal{J}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{\frac{1}{p'}} + \mathcal{C}^{\frac{1}{p'}} \right) \left(\mathcal{B}^{\frac{1}{pq'}} + \mathcal{D}^{\frac{1}{pq'}} \right), \tag{28}$$

and

$$\mathcal{I}^{1-\frac{1}{pq}} \leq \left(\mathcal{A}^{\frac{1}{qp'}} + \mathcal{C}^{\frac{1}{qp'}} \right) \left(\mathcal{B}^{\frac{1}{q'}} + \mathcal{D}^{\frac{1}{q'}} \right). \tag{29}$$

From Lemma 2, with $\lambda \geq \max\{p', q'\} - 1$, we have

$$\begin{array}{ll} \mathcal{A} = \Lambda_{p'}\, T^{1-p'}, & \mathcal{B} = \Lambda_{q'}\, T^{1-q'}, \\ \mathcal{C} = \Lambda_{p',\alpha}\, T^{1-\alpha p'}, & \mathcal{D} = \Lambda_{q',\beta}\, T^{1-\beta q'}, \end{array} \tag{30}$$

and thus (28) takes the form

$$\mathcal{J}^{1-\frac{1}{pq}} \leq K_1 \left(T^{s_1} + T^{s_2} + T^{s_3} + T^{s_4} \right), \tag{31}$$

where

$$K_1 = \max\left\{ \Lambda_{p'}^{\frac{1}{p'}} \Lambda_{q'}^{\frac{1}{pq'}}, \Lambda_{q'}^{\frac{1}{pq'}} \Lambda_{p',\alpha}^{\frac{1}{p'}}, \Lambda_{p'}^{\frac{1}{p'}} \Lambda_{q',\beta}^{\frac{1}{pq'}}, \Lambda_{p',\alpha}^{\frac{1}{p'}} \Lambda_{q',\beta}^{\frac{1}{pq'}} \right\} = \Lambda_{p'}^{\frac{1}{p'}} \Lambda_{q'}^{\frac{1}{pq'}},$$

and

$$\begin{array}{llll} s_1 & = -\left(\frac{1}{pq} + \frac{1}{p} \right), & s_2 & = -\left(\frac{1}{pq} + \frac{1}{p} \right) + 1 - \alpha = s_1 + 1 - \alpha, \\ s_3 & = -\left(\frac{1}{pq} + \frac{\beta}{p} \right), & s_4 & = -\left(\frac{1}{pq} + \frac{\beta}{p} \right) + 1 - \alpha = s_3 + 1 - \alpha. \end{array}$$

Note that $s_1 < s_3 < 0$ and $s_2 < s_4$. Thus $\mathcal{J}$ is bounded for $s_4 \leq 0$. This is equivalent to

$$1 - \frac{1}{pq} \leq \alpha + \frac{\beta}{p}. \tag{32}$$

Now, from (23),

$$v_0 \int_0^T D_{T-}^{\beta}\varphi \le \mathcal{J}^{1/q}\left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right).$$

From Lemma 1 and (30),

$$\begin{aligned} v_0 &\le K_{\beta,\lambda} T^{\beta-1}\, \mathcal{J}^{1/q}\left(\mathcal{B}^{1/q'} + \mathcal{D}^{1/q'}\right) \\ &\le K\, \mathcal{J}^{1/q}\left(T^{z_1} + T^{z_2}\right), \end{aligned} \tag{33}$$

where

$$z_1 = -2 + \beta + \frac{1}{q'} = -1 + \beta - \frac{1}{q} < 0, \qquad z_2 = -1 + \frac{1}{q'} = -\frac{1}{q} < 0, \tag{34}$$

and

$$K = \Lambda_{q'}^{\frac{1}{q'}} K_{\beta,\lambda}^{-1}.$$

When $T \to \infty$, we obtain the contradiction $0 < v_0 \le 0$.

A similar analysis could be performed by showing that $\mathcal{I}$ is bounded when

$$1 - \frac{1}{pq} \le \beta + \frac{\alpha}{q}, \tag{35}$$

which leads via (22) to the contradiction $0 < u_0 \le 0$. ∎

3.2. Remarks

REMARK 2. We can obtain a bound on the blow-up time as follows. Since $s_4 = \max_i\{s_i\}$, we have from (31)

$$\mathcal{J}^{1/q} \le (4K_1)^{\frac{p}{pq-1}}\, T^{\frac{ps_4}{pq-1}}.$$

Moreover, from (34), we have $z_1 < z_2$. It follows from (33) that

$$v_0 \le C_v\, T^s,$$

where

$$s = \frac{p\alpha + \beta}{1 - pq} < 0, \qquad C_v = 2^{1+\frac{2p}{pq-1}}\, K_1^{\frac{p}{pq-1}}\, K.$$

So, a bound on the blow-up time is given by

$$T_v = \left(\frac{v_0}{C_v}\right)^{\frac{1}{s}}.$$

A similar bound can be obtained in terms of u_0 given by

$$T_u = \left(\frac{u_0}{C_u}\right)^{\frac{1}{s}}.$$

The bound on the blow-up time is then given by the $\min\{T_u,\ T_v\}$.

Remark 3. We can obtain the same result with the less restrictive condition

$$u_0 + v_0 > 0.$$

Remark 4. When system (1) is considered, the analysis above shows that the proof goes through, if the conditions on the terms F and G

$$\int_0^\infty F(t)\,dt \geq 0, \qquad \int_0^\infty G(t)\,dt \geq 0$$

are imposed. Of course, the threshold exponents will depend on the behavior of the functions $f(t)$ and $g(t)$ at infinity.

To explain this point, let us recall the weak formulation of solutions to the system (1)

$$\int_0^T F(t)\,\varphi + \int_0^T f(t)|v|^q\,\varphi + u_0\left(1+\int_0^T D^\alpha_{T-}\varphi\right) = -\int_0^T u\,\varphi' + \int_0^T u\,D^\alpha_{T-}\varphi,$$

$$\int_0^T G(t)\,\varphi + \int_0^T g(t)|u|^p\,\varphi + v_0\left(1+\int_0^T D^\beta_{T-}\varphi\right) = -\int_0^T v\,\varphi' + \int_0^T v\,D^\beta_{T-}\varphi.$$

If, for example,

$$f(t) \geq t^{l_1}, \quad t \gg 1 \quad \text{for} \quad l_1 > 0,$$

$$f(t) \geq t^{l_2}, \quad t \gg 1 \quad \text{for} \quad l_2 > 0,$$

then one has to use the estimates

$$\int_0^T |u|\,|\varphi'| \leq \left(\int_0^T |u|^p\,\varphi g(t)\right)^{1/p}\left(\int_0^T (g(t)\varphi)^{-p'/p}\,\varphi\right)^{1/p},$$

$$\int_0^T |u|\,|D^\alpha_{T-}\varphi| \leq \left(\int_0^T |u|^p\,\varphi g(t)\right)^{1/p}\left(\int_0^T (g(t)\varphi)^{-p'/p}\,|D^\alpha_{T-}\,\varphi|^{p'}\right)^{1/p'},$$

$$\int_0^T |v|\,|\varphi'| \leq \left(\int_0^T |v|^q\,f(t)\varphi\right)^{1/q}\left(\int_0^T (f(t)\,\varphi)^{-q'/q}\,|\varphi'|^{q'}\right)^{1/q'},$$

$$\int_0^T |v|\,|D^\beta_{T-}\varphi| \leq \left(\int_0^T |v|^q\,\varphi\,f(t)\right)^{1/q}\left(\int_0^T (f(t)\varphi)^{-q'/q}\,|D^\beta_{T-}\,\varphi|^{q'}\right)^{1/q'}.$$

When the scaling argument is used as before to conclude, it appears that the behavior of $f(t)$ and $g(t)$ will have an influence on the exponents.

Let us pass now to the second case.

4. The case $0 < p, q \leq 1$

In this case because the nonlinearity is sub-linear, as it is expected the solutions are global.

THEOREM 2. *Let* $u_0 > 0$, $v_0 > 0$, *and* $0 < p, q \leq 1$. *Then any solution to system (14) emerging from* (u_0, v_0) *is global.*

P r o o f. Integrating the first equation of system (14) from ε to t, we obtain

$$u(t) + \int_\varepsilon^t D_{0+}^\alpha (u - u_0) = u(\varepsilon) + \int_\varepsilon^t v^p$$

or

$$u(t) + \int_0^t \frac{u(\tau)}{(t-\tau)^\alpha} - \int_0^\varepsilon \frac{u(\tau)}{(\varepsilon-\tau)^\alpha} = u(\varepsilon) + \frac{u_0}{1-\alpha} t^{1-\alpha} - \frac{u_0}{1-\alpha} \varepsilon^{1-\alpha} + \int_\varepsilon^t v^p,$$

which leads to the estimate

$$u(t) \leq \int_0^\varepsilon \frac{u(\tau)}{(\varepsilon-\tau)^\alpha} + u(\varepsilon) + \frac{u_0}{1-\alpha} t^{1-\alpha} + \int_\varepsilon^t v^p.$$

If $|u|_{\infty,T} := \max_{0 \leq t \leq T} u(t)$, for any $T > 0$, then we have the estimate

$$|u|_{\infty,T} \leq \frac{\varepsilon^{1-\alpha}}{1-\alpha} |u|_{\infty,T} + u(\varepsilon) + \frac{u_0}{1-\alpha} t^{1-\alpha} + t|v|^q_{\infty,T}. \tag{36}$$

Similarly, we obtain, via the second equation of system (14), the estimate

$$|v|_{\infty,T} \leq \frac{\varepsilon^{1-\beta}}{1-\alpha} |v|_{\infty,T} + v(\varepsilon) + \frac{v_0}{1-\beta} t^{1-\beta} + t|u|^p_{\infty,T}\,. \tag{37}$$

Inequalities (36) and (37) can be arranged in the form

$$\left\{ \begin{array}{lcl} \left(1 - \frac{\varepsilon^{1-\alpha}}{1-\alpha}\right) |u|_{\infty,T} & \leq & u(\varepsilon) + \frac{u_0}{1-\alpha} t^{1-\alpha} + t|v|^q_{\infty,T} \\ \left(1 - \frac{\varepsilon^{1-\beta}}{1-\beta}\right) |v|_{\infty,T} & \leq & v(\varepsilon) + \frac{v_0}{1-\beta} t^{1-\beta} + t|u|^p_{\infty,T}\,. \end{array} \right. \tag{38}$$

As $0 \leq p \leq 1$ and $0 \leq q \leq 1$, the system (38) allows to obtain the a priori estimates

$$|u|_{\infty,T} \leq A(t), \quad |v|_{\infty,T} \leq B(t)$$

with two continuous functions $A(t)$ and $B(t)$ that are easy to compute. Whereupon, any solution emerging from the initial data (u_0, v_0) is global.

However, in the case where $0 < p < 1, 0 < q < 1$, one may expect extinction of any solution, i.e., there exists a time $t_* > 0$ such that $u(t) \equiv 0$ and $v(t) \equiv 0$ for any time $t \geq t_*$.

5. The case $1 < \alpha, \beta < 2$

5.1. The main result

We set $\alpha = 1 + \widetilde{\alpha}$ and $\beta = 1 + \widetilde{\beta}$ with $0 < \widetilde{\alpha}, \widetilde{\beta} < 1$.

The main result of this section is the following

THEOREM 3. *Let $1 < p, q$, $1 < \alpha, \beta < 2$ and $u_0 > 0$, $v_0 > 0$, then the system (14) subject to (2) and $u'(0) = u_{1,0} \geq 0, v'(0) = v_{1,0} \geq 0$ admits no global solutions.*

P r o o f. The weak formulation in this case reads:

$$\begin{aligned}&\int_0^T |v|^q \varphi + u_0 \left\{1 + \int_0^T D_{T-}^{1+\widetilde{\alpha}} \varphi(t)\right\} + u_{1,0} \int_0^T t \cdot D_{T-}^{1+\widetilde{\alpha}} \varphi(t) \\ &= \int_0^T u(t)\, D_{T-}^{1+\widetilde{\alpha}} \varphi(t) - \int_0^T u\varphi', \end{aligned} \tag{39}$$

$$\begin{aligned}&\int_0^T |u|^p \varphi + v_0 \left\{1 + \int_0^T D_{T-}^{1+\widetilde{\beta}} \varphi(t)\right\} + v_{1,0} \int_0^T t \cdot D_{T-}^{1+\widetilde{\beta}} \varphi(t) \\ &= \int_0^T v(t)\, D_{T-}^{1+\widetilde{\beta}} \varphi(t) - \int_0^T v\varphi'. \end{aligned} \tag{40}$$

With the choice of the test function φ as in (8), we clearly have

$$\int_0^T D_{T-}^{1+\widetilde{\alpha}} \varphi(t) = -\int_0^T D \cdot D_{T-}^{\widetilde{\alpha}} \varphi(t) = -D_{T-}^{\widetilde{\alpha}} \varphi(t)\big|_0^T = \left(D_{T-}^{\widetilde{\alpha}} \varphi\right)(0).$$

Recall that

$$D_{T-}^{\widetilde{\alpha}} \varphi(t) = \frac{\lambda \Gamma(\lambda - \widetilde{\alpha})}{\Gamma(1 + \lambda - 2\widetilde{\alpha})} T^{-\lambda} \, (T-t)^{\lambda - \widetilde{\alpha}}.$$

So, in particular, $\left(D_{T-}^{\widetilde{\alpha}} \varphi\right)(0) = \frac{\lambda \Gamma(\lambda - \widetilde{\alpha})}{\Gamma(1+\lambda-2\widetilde{\alpha})} T^{-\widetilde{\alpha}}$.

On the other side,

$$\int_0^T t \cdot D_{T-}^{1+\widetilde{\alpha}} \varphi(t) = -\int_0^T t \cdot D \cdot D_{T-}^{\widetilde{\alpha}} \varphi(t) = \int_0^T D_{T-}^{\widetilde{\alpha}} \varphi(t) \geq 0.$$

Now, for example, (39) can be rewritten in the form

$$\begin{aligned}&\int_0^T |v|^q\varphi + u_0\Big(1+\frac{\lambda\Gamma(\lambda-\widetilde{\alpha})}{\Gamma(1+\lambda-2\widetilde{\alpha})}T^{-\widetilde{\alpha}}\Big) + u_{1,0}\,K_{\widetilde{\alpha},\lambda}T^{1-\widetilde{\alpha}}\\ &= \int_0^T u(t)\,D^{1+\widetilde{\alpha}}_{T^-}\varphi(t) - \int_0^T u\varphi'.\end{aligned} \tag{41}$$

Following the same lines as in the first part, we obtain the estimate

$$\begin{aligned}&\int_0^T |v|^q\varphi + u_0\left(1+\frac{\lambda\Gamma(\lambda-\widetilde{\alpha})}{\Gamma(1+\lambda-2\widetilde{\alpha})}T^{-\widetilde{\alpha}}\right) + u_{1,0}\,K_{\widetilde{\alpha},\lambda}T^{1-\widetilde{\alpha}}\\ &\le \left(\int_0^T |u|^p\varphi\right)^{1/p}\left(C_1T^{1-p'} + C_2T^{1-(1+\widetilde{\alpha})p'}\right) \le C\mathcal{I}\mathcal{B},\end{aligned} \tag{42}$$

where $\mathcal{B} = T^{1-p'}(1+T^{-\widetilde{\alpha}p'})$, C_1, C_2, are constants and $C := \max\{C_1, C_2\}$.

If we set

$$\mathcal{D} = T^{1-q'}\left(1+T^{-\widetilde{\beta}q'}\right),$$

we may then write the inequality (C is a constant that may change from line to line)

$$\mathcal{J}^q + \mathbf{A} \le C\,\mathcal{B}\mathcal{I}.$$

with a clear meaning of $\mathbf{A}$.

We can obtain a similar inequality via (40)

$$\mathcal{I}^p + \mathbf{C} \le C\,\mathcal{D}\mathcal{J}.$$

with a clear meaning of $\mathbf{C}$.

So,

$$\begin{cases}\mathcal{J}^q + \mathbf{A} \le & C\,\mathcal{B}\mathcal{I}\\ \mathcal{I}^p + \mathbf{C} \le & C\,\mathcal{D}\mathcal{J}\end{cases}$$

from which we can easily obtain the estimate

$$\mathcal{J}^{q-\frac{1}{p}} \le C\,T^{1-p'+\frac{1-q'}{q}}\left(1+T^{-\widetilde{\alpha}p'}\right)\left(1+T^{-\widetilde{\beta}q'}\right)^{1/q}$$

where $C > 0$ os a constant. The requirement $p > 1$, $q > 1$ leads to a contradiction when we let $T \to +\infty$. ■

5.2. Remarks

Remark 5. The intermediate case $1 < \alpha < 2$ and $0 < \beta < 1$ can be handled in the same manner as the cases treated here. We refrain to give the details.

Remark 6. Let us mention that our analysis may be used to tackle the more general system

$$\begin{cases} \sum_{i=1}^{i=N} D_{0+}^{\alpha_i}(u^{m_i(t)} - u_0^{m_i}) + \sum_{i=1}^{i=M} D_{0+}^{\delta_i}(v^{l_i(t)} - v_0^{l_i}) & = & f_1(t,|u|,|v|), \\ \sum_{i=1}^{i=K} D_{0+}^{\beta_i}(v^{n_i(t)} - v_0^{n_i}) + \sum_{i=1}^{i=L} D_{0+}^{\sigma_i}(u^{k_i(t)} - u_0^{k_i}) & = & f_2(t,|u|,|v|), \end{cases}$$

$t > 0$, where $0 < \alpha_i, \beta_i, \delta_i, \sigma_i$, $0 < m_1 \le m_i(t) \le m_2 < \infty$, $0 < n_1 \le n_i(t) \le n_2 < \infty$, $0 < l_1 \le l_i(t) \le l_2 < \infty$, $0 < k_1 \le k_i(t) \le k_2 < \infty$, and with either

$$\begin{cases} f_1(t,|u|,|v|) \ge c_1 t^{\vartheta_1}|u|^{p_1} + c_2 t^{\vartheta_2}|v|^{p_2}, \\ f_2(t,|u|,|v|) \ge c_3 t^{\vartheta_3}|u|^{p_3} + c_4 t^{\vartheta_4}|v|^{p_4}, \end{cases}$$

for $t > 0$, where $0 < c_1, c_2, c_3, c_4,\ 0 < \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, 1 < p_1, p_2, p_3, p_4$, or

$$\begin{cases} f_1(t,|u|,|v|) \ge d_1 t^{\vartheta_1}|u|^{p_1}\,|v|^{p_2}, \\ f_2(t,|u|,|v|) \ge d_2 t^{\vartheta_2}|u|^{p_3}\,|v|^{p_4}. \end{cases}$$

with $0 < d_1, d_2, \vartheta_1, \vartheta_2, 1 < p_1, p_2, p_3, p_4$.

Remark 7. Let us stress again that the study of systems containing fractional derivatives can be drastically different from that of systems of ordinary differential equations; here is an example: it is easy to see that the system

$$\begin{cases} u'(t) = & u^{p_1}v^{q_2} \quad t > 0, \\ v'(t) = & u^{p_2}v^{q_1} \quad t > 0, \end{cases}$$

$$u(0) = u_0 > 0, \quad v(0) = u_0 > 0,$$

can be decoupled into

$$\begin{cases} u'(t) = & u^{p_1}\left(\frac{r_2}{r_1}u^{r_1} - r_2 C_0\right)^{q_2/r_2} \quad t > 0, \\ v'(t) = & v^{p_2}\left(\frac{r_1}{r_2}v^{r_2} + r_1 C_0\right)^{q_1/r_1} \quad t > 0, \end{cases}$$

where we have set

$$\frac{u^{r_1}}{r_1} - \frac{v^{r_2}}{r_2} = C_0 = \frac{u_0^{r_1}}{r_1} - \frac{v_0^{r_2}}{r_2},\ t > 0,$$

that is easy to handle. However, we are unaware of a method that can decouple systems of nonlinear fractional differential equations.

6. Numerical analysis

For the numerical treatment of the system, we write (14) in the form

$$\begin{cases} u'(t) &= -D^{\alpha}_{0+}(u-u_0)+|v|^q, \quad t>0, \\ v'(t) &= -D^{\beta}_{0+}(v-v_0)+|u|^p, \quad t>0 \end{cases} \tag{43}$$

which views $D^{\alpha}_{0+}(u-u_0)$ and $D^{\beta}_{0+}(v-v_0)$ as perturbations.

Next we write

$$\begin{cases} u(t) &= u_0+\int_0^t \left\{-D^{\alpha}_{0+}(u-u_0)(\sigma)+|v|^q(\sigma)\right\}d\sigma, \quad t>0, \\ v(t) &= v_0+\int_0^t \left\{-D^{\beta}_{0+}(v-v_0)(\sigma)+|u|^p(\sigma)\right\}d\sigma, \quad t>0, \end{cases}$$

and use the iterative scheme

$$\begin{cases} u^{(n)}(t) &= u_0+\int_0^t \left\{-D^{\alpha}_{0+}(u^{(n-1)}-u_0)(\sigma)+|v^{(n-1)}(\sigma)|^q\right\}d\sigma, \\ v^{(n)}(t) &= v_0+\int_0^t \left\{-D^{\beta}_{0+}(v^{(n-1)}-v_0)(\sigma)+|u^{(n-1)}(\sigma)|^p\right\}d\sigma, \end{cases}$$

for $n=1,2,\ldots$, starting with $u^0=u_0, v^0=v_0$.

The figures show the blowing-up character of the solutions for various values of the parameters.

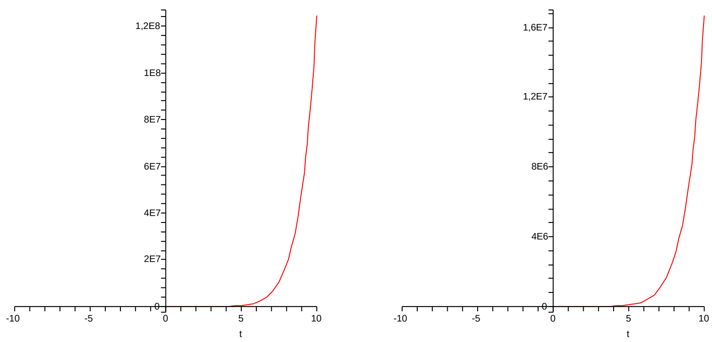

Figure 1: Curve $u^{(3)}$ at left, and curve $v^{(3)}$ at right, when $p=q=2, \alpha=\beta=\frac{1}{2}$.

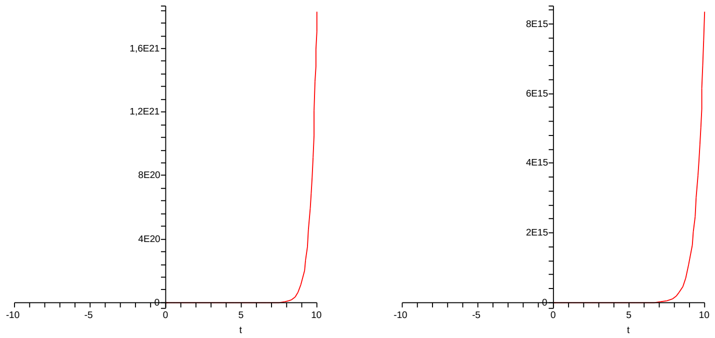

Figure 2: Curve $u^{(4)}$ at left, and curve $v^{(4)}$ at right, when $p = q = 2, \alpha = \beta = \frac{1}{2}$.

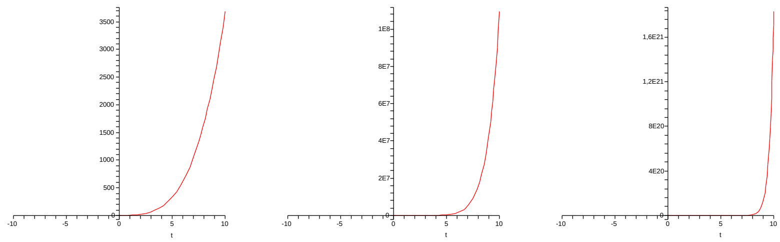

Figure 3: Curve $u^{(n)}$, $n = 2, 3, 4$ when $p = 2, q = 4, \alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$.

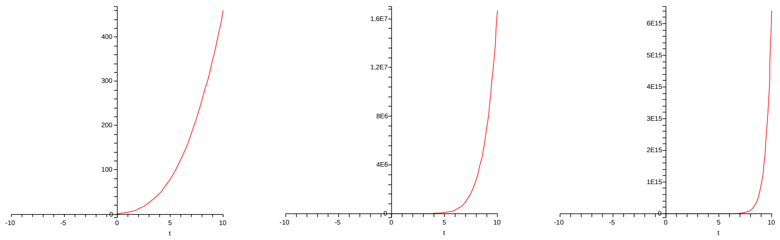

Figure 4: Curve $v^{(n)}$, $n = 2, 3, 4$ when $p = 2, q = 4, \alpha = \frac{1}{2}$ and $\beta = \frac{3}{4}$.

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